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OPTIMAL INJECTION OF COOLANT INTO THE LAMINAR BOUNDARY LAYER OF A COMPRESSIBLE GAS†

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The problem of minimizing the convective heat flux, transmitted from the boundary layer to the surface of a body in a flow, is considered under supersonic flow conditions. The specific coolant flow rate across a porous or perforated shell is regarded as the control, and the power of the cooling system, determined using Darcy's filtration law, is regarded as the constraint. Unlike similar papers on the optimal control of a boundary layer [1–3], the equation for the conservation of energy in the heat shielding shell of the apparatus is additionally considered. This approach enables one to maintain the desired temperature of the outside of the skin by profiling the skin thickness. © 2001 Elsevier Science Ltd. All rights reserved.

In order to solve the optimal problem below, we use the following: the infinitesimal Lie–Ovsyannikov apparatus [4], the theory of Nöther invariant variational problems [5, 6], the method of Dorodnitsyn generalized integral relations [7] and, also, numerical methods. A computational experiment was carried out for the case of supersonic gas flow at zero angle of attack past a spherical cap made of 14X17H2 corrosion-resistant steel with a porosity $\Pi = 0.3$, a permeability $K_\Pi = 6.4 \times 10^{-14}$ m^2 and a thermal conductivity $\lambda = 32$ W/m K. The gain in the value of the functional compared with a constant coolant flow rate along the generatrix was 32%.

1. FORMULATION OF THE PROBLEM

We will take the equations of a laminar boundary layer on a solid of revolution, past in a flow at zero angle of attack, in the form [8]

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{dp}{dx}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)$$
$$\frac{\partial}{\partial x}(\rho ur)+\frac{\partial}{\partial y}(\rho vr)=0 \tag{1.1}$$
$$\rho\left(u\frac{\partial H}{\partial x}+v\frac{\partial H}{\partial y}\right)=\frac{1}{\Pr}\frac{\partial}{\partial y}\left(\mu\frac{\partial H}{\partial y}\right)+\left(1-\frac{1}{\Pr}\right)\frac{\partial}{\partial y}\left(\mu u\frac{\partial u}{\partial y}\right)$$
$$p=\rho RT,\qquad \mu=\mu_{eo}\tau b(\tau)$$

The x axis is directed along the generatrix of the body of axial symmetry, the y axis is perpendicular to the x axis along the direction of the outward normal, u and v are the projections of the velocity vector onto the coordinate axes, p is the pressure, $r(x)$ is the radius of the solid of revolution, $H = C_pT + u^2/2$ is the total enthalpy, C_p is the heat capacity at constant pressure, T is the gas temperature, R is the gas constant, Pr is the Prandtl number, $b(\tau)$ is a known function of the dimensionless temperature $\tau = T/T_{eo}$, the subscript e corresponds to the gas parameters on the outer edge of the boundary layer and the subscript o corresponds to the gas parameters at the point of total stagnation of the flow.

The boundary conditions are taken in the form

$$y=0:\quad u=0,\ \ v=(m/\rho)_w,\quad H=H_w(x)=C_pT_w(x)$$
$$y\to\infty:\quad u\to U_e(x),\quad H\to H_e(x) \tag{1.2}$$

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Here, $T_w(x)$ is the specified temperature of the outside of the skin (it is assumed that it is equal to the gas temperature in the boundary layer), $m_w = (\rho \upsilon)_w$ is the specific flow rate of the injected gas (of the same composition as that in the free stream) across unit area in unit of time and the subscript w corresponds to gas parameters on the wall. The power consumed by the cooling system in injecting gas through the porous or perforated surface in a segment $[0, x_k]$ is estimated, taking account of Darcy's law, by the functional

$$N = 2\pi \int_0^{x_k} r\Delta p \upsilon_w dx = 2\pi a \int_0^{x_k} r \upsilon_w^2 dx \tag{1.3}$$

where the parameter a depends on the skin thickness, the porosity and permeability of the material and on the thermophysical properties of the gas in the pores.

The following variational problem is formulated. Among the continuous controls $m(x) = (\rho \upsilon)_w$, it is required to find that control which ensures that a minimum of the amount of heat

$$Q = 2\pi \int_0^{x_k} r \left(\lambda \frac{\partial T}{\partial y} \right)_{y=0} dx \tag{1.4}$$

is transferred in unit time from the boundary layer to the surface of the solid of revolution in the flow, subject to specified constraint (1.3) on the power of the cooling system and relations (1.1) and (1.2).

Using the Stepanov–Mangler–Dorodnitsyn transforms [8, 10, 11]

$$\begin{gathered} s = \frac{1}{l^3} \int_0^x r^2 \alpha_e (1-\alpha_e^2)^{\gamma/(\gamma-1)} dx, \quad t = \frac{U_e \eta}{l\sqrt{l\nu_{eo} V_{\max}}} \\ \eta = \int_0^y \frac{(1-\alpha_e^2)^{\gamma/(\gamma-1)}}{\tau} r dy, \quad U = \frac{u}{u_e}, \quad V = \sqrt{\frac{V_{\max} l^3}{\nu_{eo}}} \frac{w}{u_e} + \frac{\dot{U}_e}{U_e} tU \\ w = (1-\alpha_e^2)^{-\gamma/(\gamma-1)} \frac{u}{r^2} \frac{\partial \eta}{\partial x} + \frac{\upsilon}{\tau r}, \quad \psi = 1 - \tau - \alpha^2 \\ V_{\max} = V_\infty \sqrt{1 + \frac{2}{(\gamma-1) M_\infty^2}}, \quad \alpha = \frac{u}{V_{\max}} \end{gathered} \tag{1.5}$$

where l is a certain characteristic dimension (for example, the radius of the sphere r_0 in the case of flow past a spherical cap), system (1.1) reduces to the form

$$\begin{gathered} U\frac{\partial U}{\partial s} + V\frac{\partial U}{\partial t} = \beta(1 - U^2 - \psi) + \frac{\partial}{\partial t}\left(b(\tau) \frac{\partial U}{\partial t} \right) \\ \frac{\partial U}{\partial s} + \frac{\partial V}{\partial t} = 0 \\ U\frac{\partial \psi}{\partial s} + V\frac{\partial \psi}{\partial t} = \frac{1}{\Pr} \frac{\partial}{\partial t}\left(b(\tau) \frac{\partial \psi}{\partial t} \right) + \alpha_e^2 \left(\frac{1}{\Pr} - 1 \right) \frac{\partial}{\partial t}\left(b(\tau) \frac{\partial U^2}{\partial t} \right) \end{gathered} \tag{1.6}$$

Boundary conditions (1.2) take the form

$$\begin{gathered} t = 0: \quad U = 0, \quad V = m/q, \quad \psi = 1 - \tau_w \\ t \to \infty: \quad U \to 1, \quad \psi \to 0 \end{gathered} \tag{1.7}$$

where

$$m(s) = \frac{(\rho \upsilon)_w}{\rho_{eo}} \sqrt{\frac{l}{\nu_{eo} V_{\max}}}, \quad q = \left(\frac{r}{l} \right) \alpha_e (1-\alpha_e^2)^{\gamma/(\gamma-1)}, \quad \beta = \frac{d\alpha_e / ds}{\alpha_e (1-\alpha_e^2)}$$

Functional (1.4) and isothermal condition (1.3) are written in the form

$$Q^* = \frac{Q\,\mathrm{Pr}}{2\pi\mu_{eo}C_pT_{eo}}\sqrt{\frac{\nu_{eo}}{l^3V_{\max}}} = -\int_0^{s_k}\left(b\frac{\partial\psi}{\partial t}\right)_{t=0}ds \tag{1.8}$$

$$N^* = \frac{N}{2\pi a\nu_{eo}lV_{\max}} = \int_0^{s_k} fm^2(1-\psi_w)^2 ds, \quad f = \frac{l}{r\alpha_e(1-\alpha_e^2)^{3\gamma/(\gamma-1)}} \tag{1.9}$$

In the new variables, the variational problem is formulated as follows: among the continuous controls $m(s)$, it is required to find that control which gives a minimum value of functional (1.8) subject to conditions (1.6)–(1.7) and isothermal condition (1.9).

The Euler–Lagrange equations for the axially symmetric case are identical in form to the Euler–Lagrange equations for the plane case [12, 13], since Eqs (1.6) are identical in form to the boundary-layer equations for the plane case. A similar assertion also holds in the case of the transversality conditions. Consequently, to solve the optimization problem one can use the approach in [12], which is based on the combined use of the theory of Nöther invariant problems and the infinitesimal apparatus of Lie and Ovsyannikov. This approach enables one to obtain (as in the plane case) an approximate similar formula for the optimum specific flow rate of the coolant in the neighbourhood of the critical point in the form

$$m(x) = \frac{\mathrm{Pr}(1-\alpha_e^2)^{2\gamma/(\gamma-1)}}{2\alpha U_0(1-\omega_0/\theta_0)^2}\left[(A_1^{(1)}+N_0B_1^{(1)})+\left(\frac{x}{x_k}\right)^{-p_1}(A_1^{(2)}+N_0B_1^{(2)})+\right.$$
$$\left.+\left(\frac{x}{x_k}\right)^{-p_2}(A_1^{(3)}+N_0B_1^{(3)})\right] \tag{1.10}$$

Here

$$A_1^{(1)} = \frac{\gamma^*b+s^*P_0}{p_1p_2}, \quad A_1^{(2)} = \frac{p_1a+\gamma^*b+s^*P_0}{p_1(p_1-p_2)}, \quad A_1^{(3)} = \frac{p_2a+\gamma^*b+s^*P_0}{p_2(p_2-p_1)},$$

$$B_1^{(1)} = \frac{12\sigma^*}{p_1p_2}, \quad B_1^{(2)} = \frac{12\sigma^*+p_1s^*}{p_1(p_1-p_2)}, \quad B_1^{(3)} = \frac{12\sigma^*+p_2s^*}{p_2(p_2-p_1)}$$

$$a = \gamma^*U_0 - s^*N_0, \quad b = S_0 - 2U_0, \quad \sigma^* = s^*Q_0 - \gamma^*U_0$$

$$P_0 = \frac{l_0}{\mathrm{Pr}}\left(\frac{2m_0}{\theta_0}+\frac{\omega_0}{\theta_0}-1-\frac{2c}{\theta_0^2}\right), \quad U_0 = -\frac{\theta_0}{6}+\frac{\theta_1}{4} \tag{1.11}$$

$$N_0 = \frac{l_0}{6}\left(1-\frac{1}{\mathrm{Pr}}\right)+\frac{l_1}{12}, \quad Q_0 = \frac{2c}{\theta_0^2}, \quad S_0 = m_0 - W_0$$

$$l_0 = 4\frac{\omega_1}{\theta_1} - 3\frac{\omega_0}{\theta_0}, \quad l_1 = 4\left(\frac{\omega_0}{\theta_0}-2\frac{\omega_1}{\theta_1}\right)$$

$$W_0 = m_0 - \frac{c}{\mathrm{Pr}\,\theta_0}+\frac{l_0}{\mathrm{Pr}}Z_0, \quad c = \sqrt{\frac{\gamma-1}{\gamma}\left(1-\frac{P_\infty}{P_{eo}}\right)}, \quad Z_0 = -\frac{\theta_0}{6}+\frac{\theta_1}{3}$$

$$\frac{P_\infty}{P_{eo}} = \left(\frac{\gamma+1}{2}M_\infty^2\right)^{-\gamma/(\gamma-1)}\left(\frac{\gamma+1}{2\gamma M_\infty^2-\gamma+1}\right)^{-1/(\gamma-1)}$$

$$\gamma^* = -P_0 - \alpha_1Q_0 + \beta_1P_0, \quad S^* = W_0 - \alpha_1U_0 - \beta_1S_0$$

$$\alpha_1 = 12\left(\frac{N_0Z_0}{U_0}-M_0\right), \quad \beta_1 = -\frac{Z_0}{U_0}; \quad M_0 = l_0\left(\frac{1}{2}+\frac{1}{6\,\mathrm{Pr}}\right)$$

and p_1 and p_2 are the roots of the quadratic equation

$$p^2 + p\left[\left(\frac{S_0}{U_0} - 2\right) + 12(Q_0 - N_0)\right] + 12\left[P_0 + Q_0\left(\frac{S_0}{U_0} - 2\right)\right] = 0 \tag{1.12}$$

It should be emphasized that this equation is not equivalent to the similar equation obtained for the plane case in [12].

The constants θ_0, θ_1, ω_0, ω_1 are found from the algebraic system of equations

$$\begin{aligned}
\theta_0 &= 18m_0 - 7\theta_1 - 9\theta_0 + 10\omega_1 + 8\omega_0 - \frac{32c}{\theta_1} + \frac{34c}{\theta_0} \\
\theta_1 &= 12m_0 - 4\theta_0 - 6\theta_1 + 8\omega_1 + 4\omega_0 - \frac{16c}{\theta_1} + \frac{20c}{\theta_0} \\
\omega_1 &= 6m_0\frac{\omega_0}{\theta_0} - \omega_0 - 3\omega_1 - 4\frac{\omega_1^2}{\theta_1} + \frac{\omega_0^2}{\theta_0} + 6c\frac{\omega_0}{\theta_0^2} + \\
&+ c\left(1 + \frac{1}{\Pr}\right)\left[\frac{1}{6\theta_0}\left(4\frac{\omega_1}{\theta_1} - 3\frac{\omega_0}{\theta_0}\right) - \frac{2\omega_0}{\theta_1\theta_0}\right] - \frac{6c}{\Pr\theta_0}\left(4\frac{\omega_1}{\theta_1} - 3\frac{\omega_0}{\theta_0}\right) \\
\omega_0 &= \theta_0(1 - \tau_w)
\end{aligned} \tag{1.13}$$

which hold for the case when the viscosity depends linearly on the temperature.

In formulae (1.11) and (1.13), m_0 is the coolant flow rate, which is constant along the generatrix of the solid of revolution.

2. THE SHAPE OF THE INTERNAL WALL CONTOUR WHICH ENSURES THE REQUIRED TEMPERATURE OF THE OUTSIDE OF THE SKIN

Usually, in solving problems of non-destructive heat shielding, it is assumed that the desired wall temperature (on the gas flow side) is maintained using an appropriate, regulating flow of coolant through a porous or perforated surface. Since, in solving the variational problem considered here, the local gas flow rate has already been chosen as the control, we will select a variable wall thickness in order to ensure the required temperature of the outside of the skin. From a mathematical point of view, the equation of conservation of energy in the heat shielding shell of the apparatus has to be added to the boundary-layer equations. When account is taken of the assumptions that the medium is homogeneous and one-temperature in the case of steady flow past a sphere, the equation of conservation of energy can be written in the form [14]

$$\lambda^*\frac{\partial^2 T^*}{\partial y^2} - \left(\frac{C_p(\rho v)_w}{(1 + y/r_0)^2} - \frac{2}{r_0}\frac{\lambda^*}{1 + y/r_0}\right)\frac{\partial T^*}{\partial y} = 0 \tag{2.1}$$

$$\lambda^* = \lambda_1(1 - \Pi) + \lambda_2\Pi$$

Here, T^* is the temperature of the porous wall, λ is the thermal conductivity, ρ is the density, C_p is the specific heat capacity, r_0 is the radius of the sphere (the subscript 1 corresponds to the characteristics of the solid component of the shell and the subscript 2 corresponds to the characteristics of the coolant, which is injected into the porous material), Π is the porosity of the material, and the x and y coordinates are chosen in the same way as for the main flow.

The following boundary conditions are used for Eq: (2.1)

$$y = 0: \quad T^* = T_w; \quad y = -h, \quad T^* = T_c \tag{2.2}$$

where $h(x)$ is the wall thickness and T_c is the temperature of the coolant on the inner contour of the wall.

We will represent the condition for the heat fluxes to match on the streamlined surface in the form

$$\lambda^* \frac{\partial T^*}{\partial y} = \lambda \frac{\partial T}{\partial y} \tag{2.3}$$

The solution of Eq. (2.1) which satisfies boundary conditions (2.2) has the form

$$T^*(x, y) = T_w + (T_c - T_w)\left[1 - \exp\left(k^2 \frac{y/r_0}{1 + y/r_0}\right)\right]\left[1 - \exp\left(-k^2 \frac{h/r_0}{1 - h/r_0}\right)\right]^{-1} \tag{2.4}$$

$$k^2 = \frac{r_0 c_{p_1} (\rho v)_w}{\lambda^*}$$

Taking account of relation (2.4), equality (2.3) can be written in the form

$$-\frac{1}{f_0}\left(4\frac{\varphi_1}{f_1} - 3\frac{\varphi_0}{f_0}\right) = \frac{r}{q r_w} \Pr m\left(1 - \frac{\varphi_0}{f_0} - \tau_c\right)\left[1 - \exp\left(-m\delta \frac{h}{1 - h/r_0}\right)\right]^{-1} \tag{2.5}$$

Here

$$\delta = \frac{\rho_{eo} C_p}{\lambda^*}\sqrt{\frac{\nu_{eo} V_{\max}}{r_0}}, \quad q = \left(\frac{r}{r_0}\right)^2 \alpha_e (1 - \alpha_e^2)^{\gamma/(\gamma-1)}$$

Solving Eq. (2.5) for $h(x)$, we obtain

$$h(x) = \frac{\ln S}{r_0^{-1} \ln S - m\delta} \tag{2.6}$$

$$S = 1 + \frac{r}{q r_0} \Pr m\left(1 - \frac{\varphi_0}{f_0} - \tau_c\right)\left[\frac{1}{f_0}\left(4\frac{\varphi_1}{f_1} - 3\frac{\varphi_0}{f_0}\right)\right]^{-1}$$

Hence, after the optimal control $m(x)$ (which gives the minimum heat flux value at the specified temperature τ_w) has been found, the variation in the wall thickness $h(x)$ which ensures the temperature τ_w is determined using formula (2.6), where the functions f_0, f_1, φ_0, φ_1 are the solutions of the approximating system of the second order of approximation (a prime denotes differentiation with respect to the variable $\bar{x} = x r_0^{-1}$ [14])

$$f_0' = 18 m\bar{r} - 6\beta\left(\frac{7f_1'}{6} + \frac{9f_0}{6} - \frac{5}{3}\varphi_1 - \frac{4}{3}\varphi_0\right) - 32\frac{q}{f_1} - 34\frac{q}{f_0}$$

$$f_1' = 12 m\bar{r} - 12\beta\left(\frac{f_0}{3} + \frac{f_1}{2} - \frac{2}{3}\varphi_1 - \frac{\varphi_0}{3}\right) - 16\frac{q}{f_1} + 20\frac{q}{f_0}$$

$$\varphi_1' = 6 m\bar{r}\frac{\varphi_0}{f_0} - 6\beta\left(\frac{\varphi_0}{6} + \frac{\varphi_1}{2} - \frac{2}{3}\frac{\varphi_1^2}{f_1} - \frac{1}{6}\frac{\varphi_0^2}{f_0}\right) + 6\frac{q\varphi_0}{f_0^2} + \tag{2.7}$$

$$+6q\left(1 + \frac{1}{\Pr}\right)\left(\frac{1}{6f_0}\left(4\frac{\varphi_1}{f_1} - 3\frac{\varphi_0}{f_0}\right) - \frac{2}{3}\frac{\varphi_0}{f_0 f_1}\right) - 6\frac{q}{\Pr f_0}\left(4\frac{\varphi_1}{f_1} - 3\frac{\varphi_0}{f_0}\right) + 4q\left(\frac{1}{\Pr} - 1\right)\frac{\alpha_e^2}{f_1}$$

$$\varphi_0' = -f_0 \tau_w' + (1 - \tau_w) f_0'$$

$$\bar{r} = \frac{r}{r_0}, \quad \beta = \frac{\alpha_e'}{\alpha_e(1 - \alpha_e^2)}, \quad q = r^2 \alpha_e (1 - \alpha_e^2)^{\gamma/(\gamma-1)}$$

The initial conditions (in the case of a blunt solid of revolution) have the form

$$f(0) = f_1(0) = \varphi(0) = \varphi_1(0) = 0$$

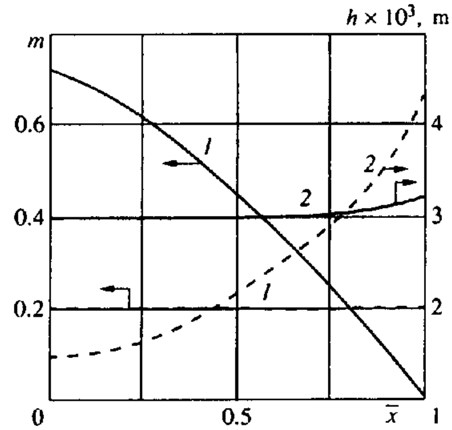

Fig. 1

3. A COMPUTATIONAL EXPERIMENT TO OPTIMIZE HEAT AND MASS TRANSFER ON A PERMEABLE SPHERE

A computational experiment was carried out for the case of supersonic flow ($M_\infty = 7$) past a sphere of radius $r_0 = 0.05$ m. The parameters of standard atmosphere corresponded to a height $H = 10^4$ m. Corrosion-resistant 14X17H2 steel with a porosity $\Pi = 0.3$, a thermal conductivity $\lambda^* = 32$ W/m K and a permeability $K_\Pi = 6.4 \times 10^{-14}$ m^2 [9] was chosen as the wall material. The dependence of the viscosity on temperature was taken as linear and $b(\tau) = 1$ [10].

The optimal injection law $m(\bar{x})$ (solid curve 1), determined using formula (1.10) for the case when $x_k = 1$, $\tau_w = 0.25$, $T_c = 273$ K and for a power corresponding to constant injection $m_0 = 0.2$ (dashed curve 1) is shown in the figure. The gain in the value of the functional compared with constant injection is approximately 32%. The wall thicknesses corresponding to the optimal control (solid curve 2) and to constant injection (dashed curve 2) are also shown. It should be noted that the wall corresponding to the optimal injection is easier to make (its thickness is almost constant) than a wall which corresponds to constant injection.

Note that, in the case of a flow past a sphere, the efficiency of porous cooling is estimated as the ratio of the integral heat flux, which is shielded because of the injection, to the gas flow rate [16]

$$J = \left[\int_0^{x_k} r q_0 dx - \int_0^{x_k} r q_w dx\right]\left[(H_{eo} - H_w)\int_0^{x_k} r(\rho v)_w dx\right]^{-1} \tag{3.1}$$

where q_0 is the local heat flux when there is no injection and q_w is the local heat flux when there is injection.

If the total flow rate of the coolant

$$\frac{P}{2\mu r_0} = \int_0^{x_k} r(\rho v)_w dx$$

is specified, then, at a specified wall temperature, the problem of maximizing functional (3.1) is equivalent to the problem of minimizing the heat flux (1.4) which has been considered earlier.

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